SPECIAL DETERMINANTS

by

FILIZ AYDINIC

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Approved by:

Leonard E. Fuller

Major Professor

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INTRODUCTION

If a particular form of an algebraical expression is lengthy, and of frequent occurence, it is desirable and of great advantage to introduce a name and symbol for it. If the form is one of a family, it is also desirable that the names and symbols indicate this relationship.

The expressions

$$a_1b_2-a_2b_1 \text{ , } a_1b_2c_3+a_3b_1c_2+a_2b_3c_1-a_3b_2c_1-a_2b_1c_3-a_1b_3c_2$$

are examples of important forms which often occur in analysis. They may be the result of eliminating variables from a system of linear homogeneous equations. Thus, elimination of x and y from the equations

$$a_1x + a_2y = 0$$
$$b_1x + b_2y = 0$$

gives as a result

$$a_1b_2 - a_2b_1 = 0 .$$

In like manner, elimination of x, y, and z from

$$a_1x + a_2y + a_3x = 0$$
$$b_1x + b_2y + b_3z = 0$$
$$c_1x + c_2y + c_3z = 0$$

gives

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 = 0 .$$

In general, eliminating the variables from a system of n linear homogeneous equations, gives as a result, the form

$$\sum \pm a_1b_2c_3 \dots k_n = 0 .$$

These forms are called Determinants. They are to be considered in this report, without reference to their origin.

Definition:

If n^2 quantities can be arranged in a square array of n rows and n columns, then the sum of all the terms that can be formed by taking the product of n quantities, one from each column and one from each row, is called the Determinant of these quantities. It is said to be of the n^{th} order. The sign of each term is determined by the number of inversions necessary to put the subscripts of the elements in the natural order of the integers when the letters are in natural order. One uses the sign + or - according as the total number of inversions is even or odd.

The ordinary notation for determinants is

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} , \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} , \dots , \begin{vmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ b_1 & b_2 & \cdot & \cdot & \cdot & b_n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ k_1 & k_2 & \cdot & \cdot & \cdot & k_n \end{vmatrix} .$$

The quantities a_1 , a_2 , etc. are called the elements of the determinant.

There are certain classes of determinants which possess special properties by virtue of some mutual relation among the elements or of some particular disposition of the same. It is the purpose of this report to present a discussion of a few of the more important of the special forms, namely, Jacobians, Wronskians, and Circulants.

The importance of the first two arises from the fact that in the application of the calculus there frequently appear determinants whose elements are differential coefficients of systems of functions. Most of the determinants originating in this manner come under one or the other of these two classes. The symbol for the ordinary derivative will also be used for the partial derivative. The content will indicate which they are.

The theorems included in this report cover a very small portion of the material obtained from the references but are complete as a short treatise on the concepts involved.

It is of utmost importance to list some of the general properties of determinants and define the terminology which will be employed in proving certain theorems later on.

General Properties:

A determinant is a linear homogeneous function of

the elements in any row or any column.

If any two columns or any two rows of a determinant are interchanged, the new determinant is equal in magnitude to the original one but has the opposide sign.

The multiplicative law of the determinants says the following: if two determinants of the same order are multiplied by each other, then the element in the pth row and qth column of the new product determinant is obtained by multiplying each element of the p^{th} row of the first determinant by the corresponding elements of the q^{th} row of the second and adding the products thus obtained.

If all elements of a row of a determinant are multiplied by the same quantity, the resulting determinant equals the product of the original determinant and the said quantity.

If two rows of a determinant are identical, the determinant is equal to zero.

If the elements in any row differ only by the same factor from the elements in any other row, the determinant is equal to zero.

Definitions:

The cofactor of any element of a determinant is found by deleting the row and the column in which the element appears and then forming the determinant of the remaining elements.

The transpose of a determinant is obtained by interchanging its rows with its columns.

A quantic is a homogeneous function of any number of variables and of any degree. A covariant is a function involving both the coefficients and the variables of a quantic such that when the quantic is subjected to a linear transformation, the same function of the new coefficients and variables is equal to the original function multiplied by a power of the determinant of the transformation.

One of the three kinds of symmetry that may occur in a determinant is symmetry with respect to the center (the point of intersection of its two diagonals). When this is true, the determinant is said to be centro-symmetric. If all the elements on every line perpendicular to the diagonal of a determinant are equal, the determinant is said to be per-symmetric or ortho-symmetric.

Some of the terms are left to be defined as they occur in the main body of the report, and one proceeds to the discussion of the four classes mentioned earlier.

JACOBIANS

Definition:

Let $u_1, \ldots, u_n$ be n functions, each of n independent variables $x_1, \ldots, x_n$. Then, the determinant $|a_{ik}|$ where $a_{ik} = \frac{du_i}{dx_k}$ is called the Jacobian of the functions $u_1, \ldots, u_n$ with respect to the variables $x_1, \ldots, x_n$. It is denoted by $\frac{d(u_1 \ldots u_n)}{d(x_1 \ldots x_n)}$ or, when there is no doubt as to the independent variables, by $J(u_1 \ldots u_n)$.

The Jacobians were first studied by a Jewish mathematician by the name of Karl Gustav Jacob Jacobi and were named after him by Professor Sylvester.

The chief properties of the Jacobians are contained in the following six theorems, the first three of which are closely related.

Theorem 1:

If the functions $u_1, \ldots, u_n$ are independent functions of $x_1, \ldots, x_n$, then,

$$\left|\frac{du_i}{dx_k}\right| \cdot \left|\frac{dx_i}{du_k}\right| = 1 .$$

Note: If $u_1, \ldots, u_n$ are independent, $x_1, \ldots, x_n$

can be expressed as functions of $u_1 , \ldots , u_n$.

Proof:

The truth of the above theorem can be shown by first writing these in determinant form and then multiplying the transpose of the first determinant by the second.

For the case $n = 2$,

$$\begin{vmatrix} \dfrac{du_1}{dx_1} & \dfrac{du_1}{dx_2} \\ \\ \dfrac{du_2}{dx_1} & \dfrac{du_2}{dx_2} \end{vmatrix} \cdot \begin{vmatrix} \dfrac{dx_1}{du_1} & \dfrac{dx_2}{du_1} \\ \\ \dfrac{dx_1}{du_2} & \dfrac{dx_2}{du_2} \end{vmatrix} = \begin{vmatrix} \dfrac{du_1}{du_1} & \dfrac{du_1}{du_2} \\ \\ \dfrac{du_2}{du_1} & \dfrac{du_2}{du_2} \end{vmatrix}$$

Since the derivative of a variable with respect to itself is equal to one, and the derivative of independent variables with respect to each other is equal to zero, this can be written as

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \ .$$

Theorem 2:

If $u^1 , \ldots , u^n$ are functions of $y^1 , \ldots , y^n$ which in turn are functions of $x^1 , \ldots , x^n$, then

$$\frac{d(u_1 \ldots u_n)}{d(y_1 \ldots y_n)} \cdot \frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)} = \frac{d(u_1 \ldots u_n)}{d(x_1 \ldots x_n)} \ .$$

Proof:

This can be shown directly by writing each Jacobian in the determinant form and performing the indicated multiplication.

Note:

$$\frac{du_1}{dx_k} = \frac{du_1}{dy_1}\frac{dy_1}{dx_k} + \frac{du_1}{dy_2}\frac{dy_2}{dx_k} + \cdots + \frac{du_1}{dy_n}\frac{dy_n}{dx_k}$$

Theorem 3:

If $u_1, \ldots, u_n$ are given only as implicit functions of $x_1, \ldots, x_n$, such that

$$\begin{aligned} F_1(u_1 \ldots u_n, x_1 \ldots x_n) &= 0 \\ F_2(u_1 \ldots u_n, x_1 \ldots x_n) &= 0 \\ \cdots\cdots\cdots\cdots &\cdots \\ F_n(u_1 \ldots u_n, x_1 \ldots x_n) &= 0 \end{aligned}$$

then

$$\frac{d(u_1 \ldots u_n)}{d(x_1 \ldots x_n)} = (-1)^n \frac{\dfrac{d(F_1 \ldots F_n)}{d(x_1 \ldots x_n)}}{\dfrac{d(F_1 \ldots F_n)}{d(u_1 \ldots u_n)}}$$

Proof:

If the i^{th} equation is differentiated with respect to

x_k, one gets

$$\frac{dF_i}{du_1}\frac{du_1}{dx_k} + \frac{dF_i}{du_2}\frac{du_2}{dx_k} + \dots + \frac{dF_i}{du_n}\frac{du_n}{dx_k} = -\frac{dF_i}{dx_k}$$

By the multiplicative rule of determinants,

$$\left|\frac{dF_i}{du_k}\right| \cdot \left|\frac{du_i}{dx_k}\right| = (-1)^n \left|\frac{dF_i}{dx_k}\right|$$

i.e.

$$\left|\frac{du_i}{dx_k}\right| = (-1)^n \frac{\left|\frac{dF_i}{dx_k}\right|}{\left|\frac{dF_i}{du_k}\right|}$$

which is another form of the statement to be proved.

Theorem 4:

If the functions $u^1, \dots, u^n$ are not independent but are functionally dependent, i.e. if $F(u^1, \dots, u^n) = 0$ then the Jacobian is equal to zero.

Proof:

When the above equation is differentiated with respect to the variables $x_1, \dots, x_n$, it follows that

$$\frac{dF}{du_1}\frac{du_1}{dx_1} + \frac{dF}{du_2}\frac{du_2}{dx_1} + \dots + \frac{dF}{du_n}\frac{du_n}{dx_1} = 0$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{dF}{du_1}\frac{du_1}{dx_n} + \frac{dF}{du_2}\frac{du_2}{dx_n} + \dots + \frac{dF}{du_n}\frac{du_n}{dx_n} = 0$$

Elimination of $\frac{dF}{du_1} \dots \frac{dF}{du_n}$ from this system gives

$$\begin{vmatrix} \frac{du_1}{dx_1} & \dots & \frac{du_n}{dx_1} \\ \dots & \dots & \dots \\ \frac{du_1}{dx_n} & & \frac{du_n}{dx_n} \end{vmatrix} = J = 0$$

The theorem proved above, together with its converse which follows as Theorem 5, is probably one of the most important properties of Jacobians as far as practical problems are concerned.

<u>Theorem 5</u>:

If the Jacobian is equal to zero, then the functions $u_1 , \dots , u_n$ are functionally dependent.

<u>Proof</u>:

$$\text{If } \frac{d(u_1 , \dots , u_n)}{d(x_1 , \dots , x_n)} = 0 \text{ , then } \frac{dG_k}{dx_k} = 0 \text{ .}$$

This is a result of the fact that the Jacobian of a set of n functions is always expressible as the product of n differential coefficients, namely,

$$\frac{d(u_1, \dots, u_n)}{d(x_1, \dots, x_n)} = \frac{dG_1}{dx_1} \frac{dG_2}{dx_2} \cdots \frac{dG_n}{dx_n}$$

where G_k is a function of $u_1, \dots, u_{k-1}, x_k, \dots, x_n$.[1] Then G_k does not involve x_k or $u_k = G_k(u_1, \dots, u_{k-1}, x_{k+1}, \dots, x_n)$ for $1 \leq k \leq n$.

If x_{k+1} is eliminated between this equation and

$$u_{k+1} = G_{k+1}(u_1, \dots, u_k, x_{k+1}, \dots, x_n)$$

one obtains

$$u_{k+1} = F_{k+1}(u_1, \dots, u_k, x_{k+2}, \dots, x_n)$$

which shows that u_n does not involve x_n.
Then,

$$u_n = F(u_1, \dots, u_{n-1})$$

which is the same as saying that $u_1, \dots, u_n$ are not independent, and the proof is complete.

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1 Sir Thomas Muir, A Treatise on the Theory of Determinants. (New York: Dover Publications, Inc, 1960), p. 637.

Theorem 6:

If the functions u_i are fractions with the same denominator so that

$$u_i = \frac{w_i}{v} \quad \text{and} \quad v^2 \frac{du_i}{dx_k} = v \frac{dw_i}{dx_k} - w_i \frac{dv}{dx_k}$$

then

$$\frac{d(u_1 \ldots u_n)}{d(x_1 \ldots x_n)} = \frac{1}{v^{n+1}} \begin{vmatrix} v & \frac{dv}{dx_1} & \cdots & \frac{dv}{dx_n} \\ w_1 & \frac{dw_1}{dx_1} & \cdots & \frac{dw_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ w_n & \frac{dw_n}{dx_1} & \cdots & \frac{dw_n}{dx_n} \end{vmatrix}$$

Proof:

The determinant for $J(u_1, \ldots, u_n)$ can also be expresses as a determinant of order n+1 in the following manner:

$$\frac{d(u_1 \ldots u_n)}{d(x_1 \ldots x_n)} = \begin{vmatrix} v & 0 & \cdots & 0 \\ w_1 & \frac{du_1}{dx_1} & \cdots & \frac{du_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ w_n & \frac{du_n}{dx_1} & \cdots & \frac{du_n}{dx_n} \end{vmatrix}$$

Using the substitution

$$v^2 \frac{du_i}{dx_k} = v \frac{dw_i}{dx_k} - w_i \frac{dv}{dx_k}$$

it follows that

$$\left|\frac{du_i}{dx_k}\right| = \frac{d(u_1 \ldots u_n)}{d(x_1 \ldots x_n)} = \frac{1}{v^{2n+1}} \begin{vmatrix} v & 0 & \ldots & 0 \\ w_1 & v \frac{dw_1}{dx_1} - w_1 \frac{dv}{dx_1} & \ldots & v \frac{dw_1}{dx_n} - w_1 \frac{dv}{dx_n} \\ \cdot & \cdot & \cdot & \cdot \\ w_n & v \frac{dw_n}{dx_1} - w_n \frac{dv}{dx_1} & \ldots & v \frac{dw_n}{dx_n} - w_n \frac{dv}{dx_n} \end{vmatrix}$$

If the first column multiplied by $\frac{dv}{dx_i}$ is added to the $(i+1)^{st}$ column for all i, one gets

$$\frac{d(u_1 \ldots u_n)}{d(x_1 \ldots x_n)} = \frac{1}{v^{2n+1}} \begin{vmatrix} v & v \frac{dv}{dx_1} & \ldots & v \frac{dv}{dx_n} \\ w_1 & v \frac{dw_1}{dx_1} & \ldots & v \frac{dw_1}{dx_n} \\ \cdot & \cdot & \cdot & \cdot \\ w_n & v \frac{dw_n}{dx_1} & \ldots & v \frac{dw_n}{dx_n} \end{vmatrix}$$

Division of each of the last n columns by v gives the desired result, and the expression on the right is called the Pre-Jacobian.

WRONSKIANS

Definition:

Let $u_1, \ldots, u_n$ be n functions of the single variable x. Then, the determinant $\left| a_k^{(i)} \right|$ where $a_k^{(i)} = \dfrac{d^{k-1}u}{dx^{k-1}}$ is called the Wronskian of the functions $u_1, \ldots u_n$ with respect to x. It is denoted by $W_x(u_1, \ldots, u_n)$.

The Polish mathematician Höene Wronski first introduces this function in connection with the expansion theorem which bears his name. Sir Thomas Muir was the first to call these functions Wronskians.

A theorem analogous to the fourth theorem proved for the Jacobians is as follows:

Theorem 1:

If the set of n functions are linearly dependent, then the Wronskian is equal to zero.

Proof:

Let the relation between the functions be

$$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0.$$

If this equation is differentiated (n-1) times with respect to x, one obtains n equations with the original one.

Elimination of the a from these equations gives $W_x(u_1 \ldots u_n)$ equal to zero.

The converse of this theorem is easily proved by the use of mathematical induction. A formal statement of the theorem and its proof follows:

Theorem 2:

If the Wronskian of a set of functions u_i is equal to zero, then the functions are connected by a linear relation with constant coefficients, that is, they are linearly dependent.

Proof:

The method of mathematical induction will be used. It follows that

$$W_x(w_2 , w_3 , \ldots , w_n) = 0$$

where $w_i = W(u_1 , u_i)$ [1] since $W_x(u_1 , \ldots , u_n) = 0$. From the definition of w_i above

$$\frac{w_i}{u_1^2} = \frac{\begin{vmatrix} u_1 & u_i \\ u_1' & u_i' \end{vmatrix}}{u_1^{\,2}} = \frac{u_1 u_i' - u_i u_1'}{u_1^{\,2}} = \left(\frac{u_i}{u_1}\right)'$$

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1 Laenas Gifford Weld, Short Course in the Theory of Determinants.(New York: MacMillan and Co.,1893), p. 212.

Assuming the theorem is true for the (n-1) functions $w_2 , \dots , w_n$, one has

$$a_2w_2 + a_3w_3 + \dots + a_nw_n = 0$$

or dividing through by u_1^2,

$$a_2\left(\frac{u_2}{u_1}\right)' + a_3\left(\frac{u_3}{u_1}\right)' + \dots + a_n\left(\frac{u_n}{u_1}\right)' = 0$$

This gives by integration and multiplication by u_1

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0$$

where a_1 is the constant of integration.
Hence the theorem is tue for (n-1) functions; it also holds for n functions. Inspection shows that it is true for two functions, therefore, it is true for n functions.

Theorem 3:

$$W_x(u_1 , \dots , u_n) = \left(\frac{dt}{dx}\right)^{n(n-1)/2} W_t(u_1 , \dots , u_n)$$

To prove this theorem, one needs to state one other general property of the determinants:

If a determinant B is obtained from a determinant A by adding a multiple of a row or column to another row or column, then the value of B is equal to the value of A.

Proof:

If the independent variables are changed, namely, the x's in each element of the left hand member, the new determinant thus obtained will contain u's and t's only. If one then applies to this new determinant the rule defined in the introduction for multiplication of determinants by a constant and the rule stated above, one readily obtains the result on the right hand side.

CIRCULANTS

Definition:

A determinant such that any row is obtained from the preceeding row by passing the last element over the others to the first place is called a Circulant. The Circulant

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ a_2 & a_3 & \cdots & a_1 \end{vmatrix}$$

whose first row is a_1 , a_2, ... , a_n is denoted by $C(a_1 , \ldots , a_n)$.

Note:

It is more convenient to define a Circulant as the determinant such that any row is obtained from the preceeding row by passing the first element over the others to the last place. The Circulant

$$\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_n & a_1 & \cdots & a_{n-1} \end{vmatrix}$$

is denoted by $C'(a_1 , \ldots , a_n)$.

Since Circulants are Per-symmetric determinants, the laws that govern this type apply to Circulants as well. They

also have some properties not possessed by Per-symmetric determinants in general, and the most important of these are given in the following theorems.

Theorem 1:

The determinant $C(u_1, \ldots, u_n)$ contains as a factor $u_1 + u_2 w + u_3 w^2 + \ldots + u_n w^{n-1}$ (where w is the root of the equation $x^n = 1$).

Proof:

If $A_1, A_2, \ldots, A_n$ are the cofactors of the elements of the first row,

$$C = u_1 A_1 + u_2 A_2 + \ldots + u_n A_n$$
$$0 = u_1 A_2 + u_2 A_3 + \ldots + u_n A_1$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$0 = u_1 A_n + u_2 A_1 + \ldots + u_n A_{n-1}.$$

Consider the product

$$(u_1 + u_2 w + \ldots + u_n w^{n-1})(A_1 + A_2 w^{-1} + A_3 w^{-2} + \ldots + A_n w^{-n+1}).$$

Remembering that $w^{k-1} = w^{n+k-1}$, for the coefficient of w^{k-1} one finds that

$$u_k A_1 + u_{k+1} A_2 + \ldots + u_{k-1} A_n$$

When k=1, this expression is equal to the given determinant C,

but for all other values of k, it is equal to zero. Thus, C is divisible by $u_1 + u_2 w + u_3 w^2 + \dots + u_n w^{n-1}$.

The following theorem showing additional properties of Circulants involve the term Skew-circulant which will now be defined:

The determinant obtained by changing the signs of all the elements on one side of the principal diagonal of a Circulant is called a Skew-circulant. For it, the functional symbol SC may be used.

Theorem 2:

A Circulant of the $2n^{th}$ order is expressible as the product of two determinants of the n^{th} order, a Circulant and a Skew-circulant.

Proof:

Consider $C(a_1, a_2, \dots, a_{2n})$ and reverse the order of the last n rows and then the order of the last n columns. Now one has a Centro-symmetric determinant of even order and can apply the theorem: every Centro-symmetric determinant of even order is expressible as the product of two determinants each of order n.[1]

Denote these by

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1 Sir Thomas Muir, A Treatise on the Theory of Determinants. (New York: Dover Publications, Inc., 1960), p. 364.

$$C_1(a_1 + a_{n+1} , a_2 + a_{n+2} , \dots , a_n + a_{2n})$$

and

$$SC_2(a_1 - a_{n+1} , a_2 - a_{n+2} , \dots , a_n - a_{2n})$$

where corresponding elements are $a_k + a_{n+k}$ and $a_k - a_{n+k}$. Note that C_1 is the sum of two sets of terms and SC_2 is the difference of the same sets.

Define

$$C_1 = A + B ,$$
$$SC_2 = A - B .$$

Then

$$C = C_1 \cdot SC_2 = A^2 - B^2 .$$

In like manner define

$$SC = A^2 + B^2 .$$

If in C, $a_{n+1} = a_{n+2} = a_{n+3} = \dots = a_{2n} = 0$, then

$$C(a_1, \dots ,a_n, 0 , \dots , 0) = C_1(a_1, \dots ,a_n)SC_1(a_1, \dots , a_n).$$

A similar and yet quite different theorem is the following:

<u>Theorem 3</u>:

A Circulant of $2n^{th}$ order can be expressed as a Circulant of the n^{th} order,

i.e.

$$C(a_1, \dots, a_{2n}) = C(x_1, \dots, x_n),$$

where

$$x_r = (a_1, -a_2, a_3, \dots, -a_{2n})(a_{2r-1}, a_{2r-2}, \dots, a_{2r+1}, a_{2r}).$$

Proof:

A determinant equal to $(-1)^{n(n+1)/2} C(a_1, \dots, a_{2n})$ is obtained from $C(a_1, \dots, a_{2n})$ by first placing the odd numbered rows in order and then the even numdered rows in order and changing the sign of the even numvered columns. Another determinant equal to $(-1)^{(n-1)/2} C(a_1, \dots, a_{2n})$ is obtained from this by deleting the negative signs, reversing the order of elements in each row. If one multiplies these two determinants and expresses the result as the product of two of its minors, one gets

$$(-1)^n C^2(a_1, \dots, a_{2n}) = (-1)^n C^2(x_1, \dots, x_n)$$

which is the desired result.

Theorem 4:

A Skew-circulant of odd order is expressible as a Circulant

i.e.

$$SC(a_1, \dots, a_{2n+1}) = C(a_1, -a_2, a_3, -a_4, \dots, a_{2n+1}).$$

Proof:

This is obtained from $SC(a_1, \ldots, a_{2n+1})$ by changing the sign of all elements in even numbered rows, and then in the even numbered columns.

The fact that the product of two Circulants of the same order is expressible as a Circulant is a direct result of the multiplication law and will not be states as a formal theorem.

BLOCK CIRCULANTS

A special kind of Circulants, namely, Block-circulants almost constitute a family of special determinants themselves, and no discussion of Circulants is complete without reference to the latter.

Definition:

Determinants of the form

$$\begin{vmatrix} A & B \\ B & A \end{vmatrix}, \quad \begin{vmatrix} A & B & C \\ B & C & A \\ C & A & B \end{vmatrix}, \quad \begin{vmatrix} A & B & C & D \\ B & C & D & A \\ C & D & A & B \\ D & A & B & C \end{vmatrix}, \ \ldots$$

where A, B, ... , are square arrays of any order n, are

called the Block-circulants. The blocks circulate in the same manner as the elements in the regular circulants.

Theorem 1:

The Block-circulant of n general m-line arrays is expressible as the product of two determinants, one of which may be expressed in per-symmetric form.

Proof:

If $A = |a_{1m}|$, $B = |b_{1m}|$, etc., and if all the columns are added in the symbolic form, the determinant of order $n \cdot m$

$$\Delta = \begin{vmatrix} A & B & C & \cdot & \cdot & \cdot & \cdot \\ B & C & D & \cdot & \cdot & \cdot & \cdot \\ C & D & E & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

has as a factor the determinant of the matrix $\| A+B+C+\ldots \|$. This is the determinant having for its r^{th} column the elements formed by adding the corresponding elements of the r^{th} columns of A , B , C , ... The other factor is obviously

$$\begin{vmatrix} C-B & D-C & E-D & \cdots \\ D-C & E-D & F-E & \cdots \\ E-D & F-E & G-F & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

which is per-symmetric in block notation.

It is a result of this theorem that the Block-circulant of two m-line arrays is expressible as the product of two m-line determinants.

i.e.

if n=2, then

$$\Delta = \begin{vmatrix} A & B \\ B & A \end{vmatrix} = (A - B) = (A + B)\cdot(A - B)$$

At the end of the discussion of ordinary Circulants, it was stated that the product of two Circulants is a Circulant. An analogous theorem is the following:

<u>Theorem 2</u>:

The product of two Block-circulants is homogenetic, that is, the product is one of the same form as the factors.

<u>Proof</u>:

The product in symbolic form establishes the truth of the theorem. Thus,

$$\begin{vmatrix} A & B & C & \ldots \\ B & C & D & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \begin{vmatrix} A' & B' & C' & \ldots \\ B' & C' & D' & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = \begin{vmatrix} AA'+BB'+CC'+\ldots & AB'+BC'+CD'+\ldots \\ BA'+CB'+DC'+\ldots & BB'+CC'+DD'+\ldots \\ \ldots\ldots & \ldots\ldots \end{vmatrix}$$

and the symmetry is with respect to the secondary diagonal.

In theorem 1 it was shown that the determinant of the matrix $\| A+B+C+\ldots \|$ is a factor of Δ. Let one suppose that n=3 and perform the operation $c_1 + \alpha c_2 + \alpha^2 c_3$ on

$$\Delta = \begin{vmatrix} A & B & C \\ B & C & A \\ C & B & A \end{vmatrix}$$

that is, add to the first column α times the second column plus α^2 times the third column where α is a cube root of unity and where $\alpha \cdot A$ means that each element of A is to be multiplied by α.

Then

$$\Delta = \begin{vmatrix} A + \alpha B + \alpha^2 C & B & C \\ B + \alpha C + \alpha^2 A & C & A \\ C + \alpha A + \alpha^2 B & A & B \end{vmatrix} = (A + \alpha B + \alpha^2 C) \cdot \begin{vmatrix} 1 & B & C \\ \alpha^2 & C & A \\ \alpha & A & B \end{vmatrix}$$

$$= (A + \alpha B + \alpha^2 C) \cdot \begin{vmatrix} C - \alpha A & A - \alpha B \\ A - \alpha B & B - \alpha C \end{vmatrix}$$

which shows that the determinant of matrix $\| A + \alpha B + \alpha^2 C \|$ is also a factor. Therefore,

$$\Delta = -(A + B + C) \cdot (A + \alpha B + \alpha^2 C) \cdot (A + \alpha^2 B + \alpha C) .$$

If the determinant of $\| A + \alpha B + \alpha^2 C \|$ is expanded as the sum of determinants with monomial elements, one gets

$P + \alpha Q + \alpha^2 R$, where P represents all the terms independent of α, Q all those containing α as a factor, and R all those containing α^2 as a factor.

Therefore,

$$\Delta = -\,(P + Q + R)\cdot(P + \alpha Q + \alpha^2 R)\cdot(P + \alpha^2 Q + R)$$

or

$$\Delta = C(P\ ,\ R\ ,\ Q)$$

The last theorem to be included in this section has an analogous proof and is as follows:

<u>Theorem 3</u>:

In general, the Block-circulant Δ formed from n m-line general determinants is equal to the product of n determinants of order m.

i.e.

$$\Delta = h \sum_{k=0}^{n-1} \| A + \alpha^k B + \alpha^{2k} C + \dots + \alpha^{(n-1)k} L \| ,$$

where $h = (-1)^{(n-1)(n-2)/2}$.

CONCLUSION

The choice of the determinants that have been considered in this paper was mostly the result of the writer's realization of her total unfamiliarity with them even though they were frequently referred to in her earlier course study of mathematics. Another factor that resulted in her exclusion of other, maybe even more important symmetric determinants was the fact that, they were studied by a former student.

Emphasis was given to properties and theorems already developed by the scholars listed in the bibliography since limited space and time elements restricted independent work on the part of the writer.

The interested reader may find the bibliography most helpful should he be interested in other forms of special determinants, which by the way, are amazingly large in number. Specially, the works of Sir Thomas Muir revised in _A Treatise on the Theory of Determinants_ by William H. Metzler is very enlightening and easy to follow.

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SPECIAL DETERMINANTS

by

FILIZ AYDINIC

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Major Professor

The purpose of this report is to present a short discussion of a few of the determinants of special form. The first two classes introduced are the Jacobians and the Wronskians whose speciality is due to the nature of their elements. The third, and last, group of special determinants included in this report are the Circulants, and their speciality is by virtue of the arrangement of their elements.

The methods and procedures employed in gathering the data mainly consists of reading the publications listed in the bibliography. Most of these books, especially Sir Muir's works, contain a very through study of the determinants mentioned above. The proofs of a great number of the theorems to be discussed later on followed identical steps in most of the references referred to. This, sometimes proved to be a handicap since some of such theorems called for additional steps and detailed enlargements for clarity, and the writer had nothing to rely on but her intuition and trial-and-error methods.

The main body of the report contains a definition of the special kind of determinant at hand and a presentetion of a few of their most important properties in the form of theorems. Special effort was made to accomplish this task without referring the reader to outside material.

As the careful reader will readily see, the Jacobians and the Wronskians possess similar properties, but the proofs of their turth vary considerably. The most outstanding fact about the Circulants and the Block-circulants is their close relationship to other forms of symmetric determinants. Once again, these very similar two classes, naturally, obey analogous rules some of which call for different methods of proof.

It is the belief of the writer that a careful reading and understanding of the introduction, which mainly consists of defining the terminology to be employed and the general properties of determinants, is most helpful and sufficient for the understanding of this report.